INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, Second Semester, 2011-12
Statistics - II, Semesteral Examination, May 9, 2012

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu_0, \sigma^2)$ where μ_0 is known and σ^2 is unknown. Consider testing at level α

$$H_0 : \sigma^2 \leq \sigma_0^2 \text{ versus } H_1 : \sigma^2 > \sigma_0^2,$$

where $\sigma_0^2 > 0$ is known.
(a) Show that the conditions required for the existence of UMP test are satisfied here.
(b) Derive UMP test of level α.
(c) Consider the test which rejects H_0 whenever $\sum_{i=1}^n (X_i - \bar{X})^2 > C$ where $C > 0$ is such that $\sup_{\sigma^2 \leq \sigma_0^2} P(\sum_{i=1}^n (X_i - \bar{X})^2 > C) = \alpha$. Show that this test is not UMP test of level α. **[10]**

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with density $f(x|\lambda) = \lambda \exp(-\lambda x)$, $x > 0$, where $\lambda > 0$ is unknown.
For testing $H_0 : \lambda = 1$ versus $H_1 : \lambda \neq 1$,
find the generalized likelihood ratio test at the significance level α. **[8]**

3. Consider a trial which ends up in 'Success' with probability θ or 'Failure' with probability $1 - \theta$, $0 < \theta < 1$. Let X denote the number of independent trials required to obtain the first 'Success'. Let $X_1, \ldots, X_n$ be a random sample from the distribution of X.
(a) Find the maximum likelihood estimator $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$ of θ.
(b) Show that $\hat{\theta}_n$ is a consistent estimator of θ.
(c) Find the asymptotic distribution of $\hat{\theta}_n$. **[10]**

4. The weekly number of fires, X, in a town has the $Poisson(\theta)$ distribution. The numbers of fires observed for five weekly periods were 0, 1, 1, 0, 0. Assume that the prior distribution on θ is
$\pi(\theta) \propto \theta \exp(-10\theta) I_{(0,\infty)}(\theta)$.
(a) Derive the posterior distribution θ given the data.
(b) Find the highest posterior density estimate of θ. Compare this with the maximum likelihood estimate of θ.
(c) Find the posterior mean and posterior standard deviation of θ. **[12]**

5. Consider a regular model $\{f(x|\theta), \theta \in \Theta \subseteq \mathcal{R}^k\}$. Suppose we have a random sample of size $n > k$, and T is the minimal sufficient statistic for θ.
(a) Show that the maximum likelihood estimator, if it exists, depends on the data through T only.
(b) Show that any Bayes estimator depends on the data through T only.
(c) Give an example where method of moments estimators of θ exist, but none of them is a functions of T. **[10]**